TCC Week 7.

Linear and nonlinear problems in bounded domains.

We want to look at:

1) Hardy inequalities $\int_\Omega |\nabla\varphi|^2 \geq \mu \int_\Omega \frac{\varphi^2}{\delta_{\partial\Omega}^2}$

2) Hardy operator $-\Delta - \frac{\mu}{\delta_{\partial\Omega}^2}$

3) Nonlinear problem $-\Delta u - \frac{\mu}{\delta_{\partial\Omega}^2} u + u^p = 0$ in Ω

Ω – bounded smooth domain

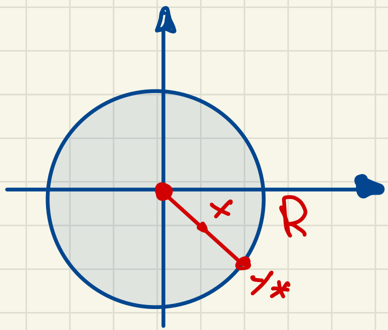
1) Hardy inequalities with distance to the boundary

Ω - Bounded open C^2-domain in $\mathbb{R}^N$, $N \geq 2$

$\partial\Omega$ is a C^2-manifold

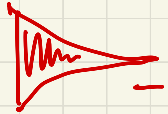
$$\delta_\Omega(x) = \mathrm{dist}(x, \partial\Omega) = \min_{y \in \partial\Omega} |x - y| \quad (x \in \Omega)$$

Example: $\delta_{B_R}(x) = |R - x|$ - nonsmooth!



But δ_Ω is Lipschitz, always

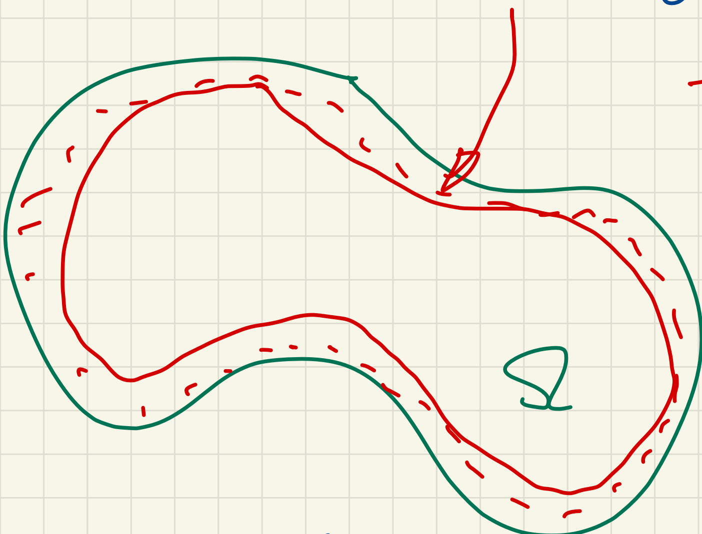
if $\partial\Omega$ is Lipschitz



- non-Lipschitz!

Notations: $\Omega_\rho = \{ x \in \Omega : \delta_\Omega(x) < \rho \}$

– δ-strip around $\partial\Omega$



$\Gamma_\rho = \{ x \in \Omega : \delta_\Omega(x) = \rho \}$

$\Rightarrow \partial\Omega_\rho = \partial\Omega \cup \Gamma_\rho$

Lemma (<u>Local</u> Hardy inequality)

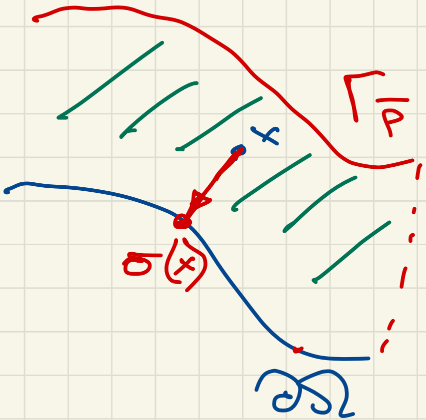
$$\int_\Omega |\nabla u|^2 \geq \frac{1}{4} \int \frac{u^2}{\delta_\Omega^2} \qquad \forall u \in C_0^\infty(\Omega_\rho), \quad \forall \rho \in (0, \bar{\rho})$$

$\frac{1}{4}$ is sharp!

Tools: If Ω is a C^2-domain then:

1) $\exists\, \bar\rho > 0$: $\delta_\Omega \in C^2(\Omega_{\bar\rho})$ and $\Gamma_{\bar\rho}$ is C^2-domain,

Γ_ε is C^2 $\forall\, \varepsilon \le \bar\rho$



2) $\forall\, x \in \Gamma_\varepsilon$ $\exists$ unique $\sigma(x) \in \partial\Omega$,

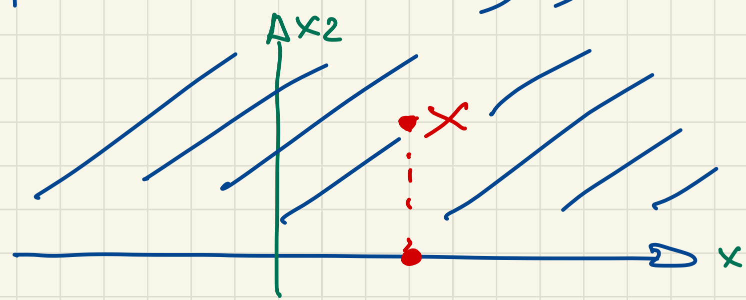
$$|x - \sigma(x)| = \delta_\Omega(x)$$

3) $|\nabla \delta_\Omega| = 1 + o(\delta_\Omega(x))$, $x \to \partial\Omega$.

"Half-plane mode e"

$\mathbb{R}^2_+$

$$\operatorname{dist}(x, \partial\mathbb{R}^2_+) = x_2$$



4) $-\Delta \delta(x) = (N-1)\,\mathcal{H}_0(\sigma(x)) + o(\delta(x)), \quad x \to \partial\Omega$

$\delta = \delta_\Omega$

$\mathcal{H}_0(y)$ – mean curvature of $\partial\Omega$ at $y \in \partial\Omega$

$\mathcal{H}_0$ is Bounded, Ω is convex $\Rightarrow \mathcal{H}_0 \geq 0$

5) $\nabla \delta^\beta = \beta \delta^{\beta-1} \nabla \delta$

Ex. $-\Delta \delta^\beta = -\beta(\beta-1)\delta^{\beta-2} \underbrace{|\nabla\delta|^2}_{\sim 1} - \beta\delta^{\beta-1}\underbrace{\Delta\delta}_{\sim \mathcal{H}_0} =$

$= -\beta(\beta-1)\delta^{\beta-2} - (N-1)\delta^{\beta-1}\mathcal{H}_0(\sigma(x)) + o(\delta^\beta)$

$-\Delta r^\gamma = -\gamma(\gamma-1) r^{\gamma-2} - \frac{N-1}{r}\gamma r^{\gamma-1}$

Summary: $-\Delta \delta^{\beta} \; \Omega \; -\beta(\beta-1)\,\delta^{\beta-2}$ in $\Omega_{\bar{\rho}}$

Almost $-\left(t^{\beta}\right)'' = -\beta(\beta-1)\,t^{\beta-2}$

"one dimensional"

Exercis: $-\Delta\Big(\underbrace{\delta^{\beta} \log^{\frac{1}{2}}\left(\tfrac{1}{\delta}\right)}_{u_*}\Big) \geq \frac{1}{4}\,\delta^{\beta-2}\underbrace{\log^{\frac{1}{2}}\left(\tfrac{1}{\delta}\right)}_{\frac{u_*}{\delta^2}}$ in $\Omega_{\bar{\rho}}$

$\Rightarrow \displaystyle\int_{\Omega} |\nabla\varphi|^2 - \frac{1}{4}\int_{\Omega}\frac{\varphi^2}{\delta^2} \geq 0 \quad \forall \varphi \in C_0^{\infty}(\Omega_{\bar{\rho}})$

by AAP-principle

— proof of local Hardy inequality

Theorem (Ancona, Marcus-Pinchover-Mizel)

If Ω is a bounded C^2-domain then:

1) $\displaystyle\int_\Omega |\nabla\varphi|^2 \geq \frac{1}{4}\int_\Omega \frac{\varphi^2}{\delta^2} \quad \forall \varphi \in C_0^\infty(\Omega_{\bar{\beta}})$ (local Hardy)

2) $\displaystyle\int_\Omega |\nabla\varphi|^2 \geq C_H(\Omega)\int_\Omega \frac{\varphi^2}{\delta^2} \quad \forall \varphi \in C_0^\infty(\Omega)$

(global Hardy)

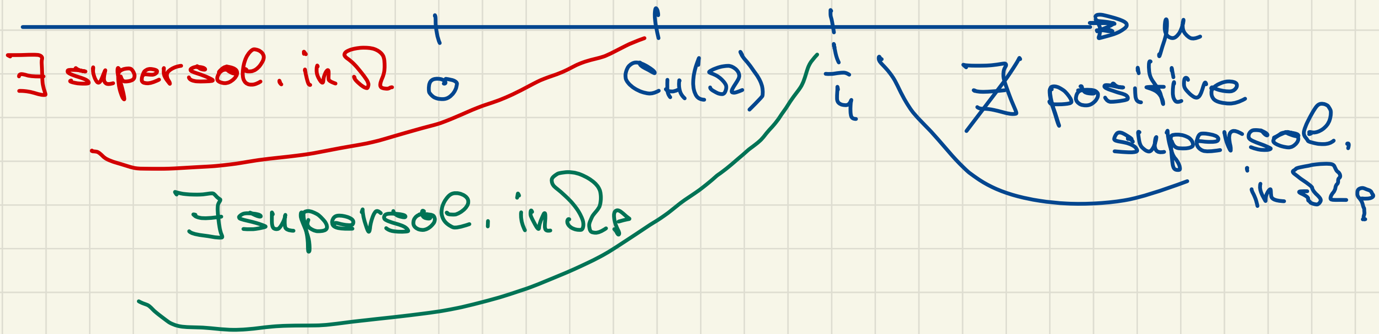
a) $0 < C_H(\Omega) \leq \frac{1}{4}$

b) $C_H(\Omega) = \frac{1}{4}$ if Ω is convex (Ancona)

c) $\forall \varepsilon \in (0, \frac{1}{4})\ \exists\, \Omega :\ C_H(\Omega) = \varepsilon$, $N \geq 3$

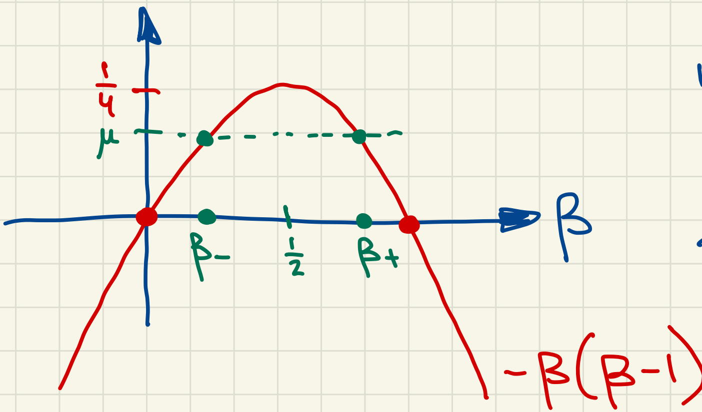
Hardy operator: $\quad -\Delta - \frac{\mu}{\delta^2}$ in Ω

and we assume $\quad \mu \leq \frac{1}{4}$



We want to understand boundary behaviour of sub and super-solutions near $\partial\Omega$ for all $\mu \leq \frac{1}{4}$.

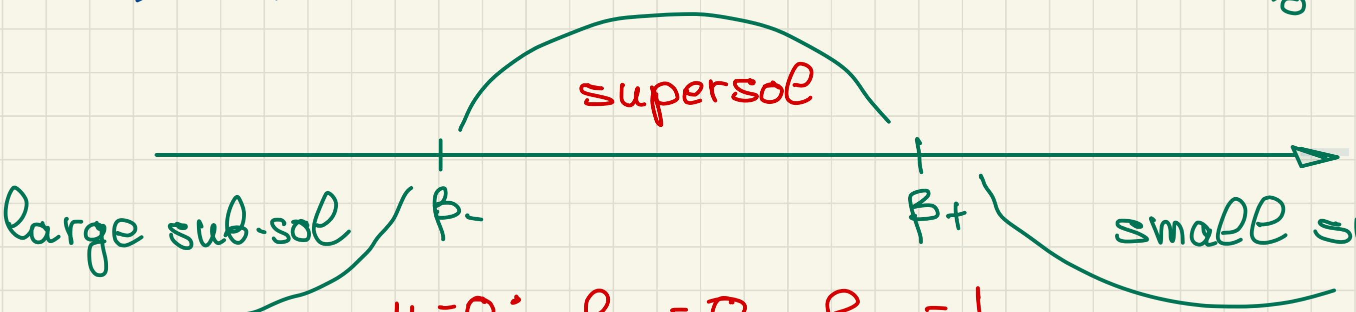
Compute $-\Delta \delta^{\beta} - \frac{\mu}{\delta^2}\delta^{\beta} = \left(-\beta(\beta-1) - \mu\right)\delta^{\beta-2} + o\left(\delta^{\beta-2}\right)$



1) $\beta \in (\beta_-, \beta_+)$
$\Rightarrow \delta^{\beta}$ is a supersol

2) $\beta \notin [\beta_-, \beta_+]$
$\Rightarrow \delta^{\beta}$ is a subsol

3) $\delta^{\beta_-}, \delta^{\beta_+}$ = "almost" solutions: $-\Delta \delta^{\beta_-} - \frac{\mu}{\delta^2}\delta^{\beta_-} \simeq \underbrace{\mathcal{H}_0}_{\text{sign-chang.}}$



$\mu = 0$: $\beta_- = 0$, $\beta_+ = 1$

$\mu = 0$: const and δ are "almost" solutions

$\mu = \frac{1}{4}$ $\delta^{\frac{1}{2}}$ and $\delta^{\frac{1}{2}} \log\left(\frac{1}{\delta}\right)$ are "almost" sol

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Agmon's trick:

1) $\delta^{\beta_+}(1-\delta^{\varepsilon})$ — supersol $\sim \delta^{\beta_+} - \delta^{\beta_+ + \varepsilon}$

$\delta^{\beta_-}(1+\delta^{\varepsilon})$ — subsol

2) $\delta^{\beta_+}(1+\delta^{\varepsilon})$ — small subsol.

$\delta^{\beta_-}(1-\delta^{\varepsilon})$ — large subsol.

in Ω_ε

By comparison,

minimal solution $u \simeq \delta^{\beta_+}$

Agmon's solution $v \simeq \delta^{\beta_-}$ $\left(\mu \leq C_H(\Omega) \right)$